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Two classes of exactly solvable quantum models with moving boundaries

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Abstract. It is shown here that the non-stationary Schrödinger equation can be solved exactly for two quantum models subject to Dirichlet boundary conditions. One of them is a modified problem of a quantum bouncer, i.e. the problem of a particle falling down in the gravitational field on a moving (oscillating) platform such as a loudspeaker. The second model is a 'cut-off oscillator' with a moving infinite potential wall and a time-dependent frequency. In both cases exact solutions are given in closed forms, easy to use. Their possible applications are also indicated. In each of the models extra coordinate- and time-dependent phase factors are generated by moving boundaries in the former case giving rise to a non-local effect in quantum mechanics.

1. Introduction

An exact solution of the Schrödinger equation with a time-dependent Hamiltonian is a very difficult task and can only be performed in very few cases. Those parabolic-type partial differential equations constitute a great challenge even for a computer approach. This is especially the case when they contain rapidly oscillating time-dependent coefficients. This happens, for example, when a particle is bound to move in some subspace of the full space and the border of the subspace oscillates according to a given function of time, say $L(t)$.

The Fermi-Ulam model [1, 2] is presumably the first one of this kind. It consists of a particle bouncing inside a one-dimensional infinite square well of oscillating width. Its classical version, known as a Fermi accelerator, has been proposed in order to explain the mechanism in which cosmic particles achieve very high energies. Though the model appeared to be too crude to solve the problem of cosmic radiation [3] it plays an important role in the theory of quantum chaos [4].

Exact solutions of the quantum counterpart of the model have been found [5, 6] for some functions $L(t)$ only. However, a modification of the Fermi accelerator [6] can be solved exactly for any function $L(t)$ and its numerical analysis [7] leads to a number of interesting results.

Our main motivation for a study of similar models follows from the fact that we know very little about the behaviour of quantum systems with time-dependent boundary conditions. Possible applications of the models to the problems of the so-called quantum chaos or to optical effects connected with moving mirrors are additionally encouraging.

2. Basic equations

The systems we want to discuss are described by the non-stationary Schrödinger equations

$$i\hbar\frac{\partial\psi}{\partial t}=\frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}+\begin{cases} mG(t)x\psi(x,t) & (1a)\\ \frac{1}{2}m\omega^2(t)x^2\psi(x,t) & (1b)\end{cases}$$

and the time-dependent boundary conditions

$$\psi(x=L(t),t)=\psi(x=\infty,t)=0. \tag{2}$$

These equations are not solvable until the moving boundaries are replaced by fixed ones. In the choice of a transformation leading to fixed boundaries one is guided only by one's experience and intuition. We take here

$$y=x-L(t). \tag{3}$$

This leads to the replacements

$$\frac{\partial}{\partial t}\to\frac{\partial}{\partial t}-\dot{L}\frac{\partial}{\partial y}\qquad\qquad \frac{\partial^2}{\partial x^2}\to\frac{\partial^2}{\partial y^2} \tag{4}$$

resulting in the new Schrödinger equations

$$i\hbar\frac{\partial\psi}{\partial t}=\frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial y^2}+i\hbar\dot{L}\frac{\partial\psi}{\partial y}+\begin{cases} mG(t)(y+L)\psi(y,t) & (5a)\\ \frac{1}{2}m\omega^2(t)(y+L)^2\psi(y,t) & (5b)\end{cases}$$

subject already to the time-independent boundary conditions

$$\psi(y=0,t)=\psi(y=\infty,t)=0. \tag{6}$$

Before specifying the two models in more detail let us note that equations (5) can be simplified if their solutions are guessed to be in the form

$$\psi(y,t)=\exp\left\{\frac{im}{2\hbar}\left[2\dot{L}y+\int_0^t\dot{L}^2\,dt_1-\begin{cases}2\int_0^t G(t_1)L(t_1)\,dt_1 & (7a)\\ \int_0^t\omega^2(t_1)L^2(t_1)\,dt_1 & (7b)\end{cases}\right]\right\}\varphi(y,t).$$

Then, we have

$$i\hbar\frac{\partial\varphi}{\partial t}=\frac{-\hbar^2}{2m}\frac{\partial^2\varphi}{\partial y^2}+\begin{cases} m[G(t)+\ddot{L}(t)]y\varphi(y,t)\equiv H_a\varphi & (8a)\\ m[\frac{1}{2}\omega^2(t)y^2+(\ddot{L}+\omega^2L)y]\varphi(y,t)\equiv H_b\varphi & (8b)\end{cases}$$

where the normalization condition reads

$$\int_{L(t)}^{\infty}\psi^*(x,t)\psi(x,t)\,dx=\int_0^{\infty}\psi^*(y,t)\psi(y,t)\,dy=1. \tag{9}$$

3. The quantum bouncer and its modification

This model is described by the equations (1a), (5a), (7a), (8a). When formulated on the entire real axis it is known as a problem of a particle moving in the field of an homogeneous time-varying force. As such, it can easily be solved with the help of any method based on the evolution of a given initial state [8, 9]. In the case of the boundary conditions (2) the methods seem to be simple to use only when the time and spatial coordinates can be separated.

In order to solve (8a) let us first note that $[H_a(t_1), H_a(t_2)] \neq 0$. The non-commutativity of H_a with itself at different times is not a serious problem for two cases considered below. We can use the following gauge transformations:

$$H_A(t) = \exp\left(-\frac{\mathrm{i}}{\hbar} S\right) H_a(t) \exp\left(\frac{\mathrm{i}}{\hbar} S\right) \tag{10}$$

$$\Phi(y, t) = \exp\left(-\frac{\mathrm{i}}{\hbar} S\right) \varphi(y, t). \tag{11}$$

If $S = S(y, t)$ and does not depend on the derivative $\partial/\partial y$, then from (8a) one obtains

$$\mathrm{i}\hbar \frac{\partial \Phi}{\partial t} = \frac{1}{2m}\left(-\mathrm{i}\hbar \frac{\partial}{\partial y} + \frac{\partial S}{\partial y}\right)^2 \Phi + [m(G + \ddot{L})y + \dot{S}]\Phi. \tag{12}$$

Use has been made of the well known Baker–Campbell–Hausdorff [10] formula and the relations

$$\begin{aligned} &[\partial/\partial y, S(y, t)] = \partial S/\partial y \\ &[\partial^2/\partial y^2, S(y, t)] = \left(\frac{\partial S}{\partial y}\frac{\partial}{\partial y} + \frac{\partial}{\partial y}\frac{\partial S}{\partial y}\right) \equiv P(y, t) \\ &[P(y, t), S(y, t)] = 2(\partial S/\partial y)^2. \end{aligned} \tag{13}$$

Now the choice of S in the form

$$S(y, t) = yQ(t) = -my \int_0^t [G(t_1) + \ddot{L}(t_1)]\, \mathrm{d}t_1 \tag{14}$$

eliminates the 'scalar potential' in (12) and the 'vector potential' depends only on t. Thus, the resulting Hamiltonian in (12) commutes with itself at different times and that is why the formal solution of (8a) reads

$$\varphi(y, t) = \exp\left[\frac{\mathrm{i}}{\hbar} Q(t)y - \mathrm{i}\alpha(t)\right] \exp\left[\mathrm{i}\beta(t)\frac{\partial^2}{\partial y^2}\right] \exp\left[\gamma(t)\frac{\partial}{\partial y}\right]\varphi(y, 0) \tag{15}$$

where

$$\begin{aligned} &\alpha(t) = \frac{1}{2m\hbar}\int_0^t Q^2(t_1)\, \mathrm{d}t_1 \\ &\beta(t) = \frac{\hbar t}{2m} \\ &\gamma(t) = (-1/m)\int_0^t Q(t_1)\, \mathrm{d}t_1 \end{aligned} \tag{16}$$

and the function $Q(t)$ is defined in (14).

A further discussion depends on a choice of the function $G(t)$ in the *a*-equations. It is possible to write (15) and hence the solutions of ($1a$) in explicit forms in two cases distinguished below.

3.1. *The bouncer*

Let us take $G(t)=g=$ constant, with g being the acceleration due to gravity, in all *a*-equations beginning from ($1a$). This case is of a special importance since its classical counterparts, both elastic and inelastic, exhibit a variety of solutions from regular to chaotic [11].

The quantum case is easily solvable if

$$L(t)\equiv L_1(t)=At^2+Bt+C \tag{17}$$

which immediately follows from the inspection of ($8a$). A separation of variables leads now to an equation for the Airy functions [12] $Ai(\ldots)$ and $Bi(\ldots)$.

To fulfil the boundary conditions (6) an acceptable solution of ($8a$) has to be in the form

$$\varphi_n(y,t)=\exp\left(-\frac{\mathrm{i}}{\hbar}\Lambda_n t\right)Ai\left[\frac{y}{\delta}-\frac{\Lambda_n}{m(g+2A)\delta}\right] \qquad g+2A>0 \tag{18}$$

where $\delta=[\hbar^2/2m^2(g+2A)]^{1/3}$ and the eigenvalues Λ_n are to be determined from the condition

$$Ai[-\Lambda_n/m(g+2A)\delta]=0. \tag{19}$$

Hence, the exact solution of ($1a$) for $L(t)$ given in (17) and obeying the conditions (2) has the final form

$$\begin{aligned}\psi_n(x,t)&=C_n\exp\left\{\frac{im}{2\hbar}\left[2\dot{L}_1(x-L_1)+\int_0^t\dot{L}_1^2\,\mathrm{d}t_1-2g\int_0^t L_1\,\mathrm{d}t_1\right]\right\}E_n(x,t)\\ E_n(x,t)&\equiv\exp\left(-\frac{\mathrm{i}}{\hbar}\Lambda_n t\right)Ai[(x-L_1(t))/\delta-\Lambda_n/m(g+2A)\delta]\end{aligned} \tag{20}$$

with the normalization constants C_n being found from (9).

The solution (20) can also be derived with the help of (15). To this end the function $F_n(y,t)$ defined by

$$F_n(y,t)\equiv\exp[\mathrm{i}\beta(t)\partial^2/\partial y^2]\exp[\gamma(t)\partial/\partial y]Ai[y/\delta-\Lambda_n/m(g+2A)\delta] \tag{21}$$

has to be calculated. The simplest way to do that consists in using an integral representation for the Airy function

$$Ai(z)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\exp[\mathrm{i}(\tfrac{1}{3}u^3+zu)]\,\mathrm{d}u \tag{22}$$

which is easily derivable from the formula 10.4.32 of [13].

Then, expanding the exponentials in (21) and using (22), we get

$$F_n(y,t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\exp\left\{\mathrm{i}\left[\tfrac{1}{3}u^3-\beta(t)(u/\delta)^2+\left(\frac{y+\gamma(t)}{\delta}-\frac{\Lambda_n}{m(g+2A)\delta}\right)u\right]\right\}\mathrm{d}u. \tag{23}$$

Changing the variable of integration in (23) to the new one

$$v=u-\beta(t)/\delta^2 \tag{24}$$

calculating $\gamma(t)$ for $L(t)$ given in (17), using the definition of $\beta(t)$ and then (22), we get

$$F_n(y,t)=\exp\left[\mathrm{i}\alpha(t)-\frac{\mathrm{i}}{\hbar}Q(t)y\right]\exp\left(-\frac{\mathrm{i}}{\hbar}\Lambda_n t\right)Ai[y/\delta-\Lambda_n/m(g+2A)\delta]. \tag{25}$$

The functions $\alpha(t)$ and $Q(t)$ have to be calculated from equations (16) and (14), respectively, with $L(t)=L_1(t)$ as defined in (17). Then, using (15), (7*a*) and (3) we obtain again (20).

3.2. *The modified bouncer*

The problem of a particle bouncing in the gravitational field can be modified by introducing an additional field such that the function $G(t)$ in all *a*-equations is given by

$$G(t)=g-\ddot{L}(t). \tag{26}$$

In this case the terms with the variables y and t separate in (8*a*) for any function $L(t)$ of the class $\mathbb{C}^2$. Hence, (8*a*) can again be replaced by two equations: that in t is a trivial one and that in y leads to the earlier-mentioned equation for the Airy functions. Thus, in analogy to (20), we have

$$\begin{aligned}\psi_n(x,t)&=D_n\exp\left\{\frac{\mathrm{i}m}{2\hbar}\left[2\dot{L}(x-L)+\int_0^t\dot{L}^2\,\mathrm{d}t_1-2\int_0^t(g-\ddot{L})L\,\mathrm{d}t_1\right]\right\}K_n(x,t)\\ K_n(x,t)&\equiv\exp\left(-\frac{\mathrm{i}}{\hbar}\lambda_n t\right)Ai[(x-L(t))/\Delta-\lambda_n/mg\Delta]\end{aligned} \tag{27}$$

where $\Delta=(\hbar^2/2m^2g)^{1/3}$, the eigenvalues λ_n are zeros of the equation $Ai(-\lambda_n/mg\Delta)=0$ and the normalization constants D_n can be obtained from (9).

Of course, the solution (27) is also derivable from the formal expression (15). The way of performing that is the same as described in section 3.1.

It is worth mentioning at this point that apart from the two cases discussed above the method based on the formal solution (15) does not generate any other exact solution of (1*a*). The source of the inconvenience are the boundary conditions (6) restricting the space of y coordinates to the real half-axis $[0,\infty)$ only. In such a case *(15) is in general an exact solution of (8a) only for stationary states of the Hamiltonian* H_a. The reader can clearly see that applying the procedure described in section 3.1 to, for example, a bouncer with $L(t)$ different from that in (17). Then starting with the initial state $\varphi(y,0)$, obeying boundary conditions on the y half-axis, we get from (15) a state at a later time t obeying (8*a*) but not the same boundary conditions. Thus, it cannot be accepted as a proper solution of (8*a*).

The cases different from those in (17) and (26) need much more sophisticated treatment than the present one and we are not going to discuss this problem here.

Our solutions (20) and (27) obviously obey conditions (2) since $Ai(\infty)=0$ and eigenvalues Λ_n and λ_n are to be chosen in such a way that $Ai=0$ for $x=L(t)$. As a final step we can easily check with the help of two formulae

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}Ai[a(t)x+b(t)]=a^2(ax+b)Ai(ax+b) \tag{28}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}Ai[a(t)x+b(t)]=\frac{1}{a}\left(\frac{\mathrm{d}a}{\mathrm{d}t}x+\frac{\mathrm{d}b}{\mathrm{d}t}\right)\frac{\mathrm{d}\,Ai(ax+b)}{\mathrm{d}x} \tag{29}$$

that (20) and (27) are solutions of (1*a*). The proof can easily be performed with the help of relations 13.6.25, 13.4.21 and 13.4.26 of [13].

4. The cut-off oscillator

This model is represented by the *b*-equations of section 2. We are able to find exact solutions of (1*b*) obeying boundary conditions (2) if a time-dependent frequency $\omega(t)$ of our oscillator and the function $L(t)$, describing the movements of its boundary, are related in the equation

$$\ddot{L}+\omega^2(t)L=0. \tag{30}$$

From (8*b*) we have now

$$\mathrm{i}\hbar\frac{\partial\varphi}{\partial t}=\frac{-\hbar^2}{2m}\frac{\partial^2\varphi}{\partial y^2}+\tfrac{1}{2}m\omega^2(t)y^2\varphi(y,t). \tag{31}$$

From a methodological point of view two cases of (31) are worth distinguishing.

Case A. $\omega(t)=\omega_0=constant>0$. As follows from (30) the boundary of the oscillator can move here according to the functions $L(t)=A_1\sin\omega_0 t$, $A_2\cos\omega_0 t$ or any of their linear combination. On the other hand, (31) can be solved with the method of separation of variables.

It is quite clear that solutions of (31) vanishing at $y=0$ and $y=\infty$ can be expressed by the well known odd Hermite polynomials H_{2n+1}. Therefore, an exact, normalized solution of (1*b*) vanishing at $x=L(t)$ and $x=\infty$ can be written at once as

$$\psi_n(x,t)=[2^{2n}\sigma_0(2n+1)!\sqrt{\pi}]^{-1/2}$$
$$\times\exp\left\{\frac{\mathrm{i}m}{2\hbar}\left[2\dot{L}(x-L)+\int_0^t\dot{L}^2\,\mathrm{d}t_1-\omega_0^2\int_0^t L^2\,\mathrm{d}t_1\right]\right\}Y_n(x,t) \tag{32}$$

$$Y_n(x,t)\equiv\exp[-\mathrm{i}\omega_0(2n+\tfrac{3}{2})t]\exp[-(x-L)^2/2\sigma_0^2]H_{2n+1}[(x-L)/\sigma_0]$$

where $\sigma_0=(\hbar/m\omega_0)^{1/2}$ and $n=0,1,2,\ldots,\infty$.

Case B. $\omega=\omega(t)>0$. If ω is an arbitrary function of time, (31) and hence (1*b*) can also be solved. Though the terms including the variables y and t do not separate we have at our disposal a number of methods [14, 15] developed for solving the problem of a harmonic oscillator with a time-dependent frequency. Due to the property of the Hermite polynomials that $H_{2n+1}(0)=0$ we can use one of the methods.

Referring the reader to the mentioned works, where several methods of solving (31) are described with full particulars, we give here just the final normalized solution of (1*b*). It reads

$$\psi_n(x,t)=[2^{2n}(2n+1)!(\pi\hbar/m\dot{\xi})^{1/2}]^{-1/2}$$
$$\times\exp\left\{\frac{\mathrm{i}m}{2\hbar}\left[2\dot{L}(x-L)+\int_0^t\dot{L}^2\,\mathrm{d}t_1-\int_0^t(\omega L)^2\,\mathrm{d}t_1\right]\right\}$$
$$\times\exp[-\mathrm{i}(2n+\tfrac{3}{2})\xi(t)]\exp\left[\frac{\mathrm{i}m}{2\hbar}\left(\frac{\dot{T}}{T}+\mathrm{i}\dot{\xi}\right)(x-L)^2\right]$$
$$\times H_{2n+1}[(x-L)(m\dot{\xi}/\hbar)^{1/2}] \tag{33}$$

where the auxiliary functions $T(t)$ and $\xi(t)$ obey the equations

$$\ddot{T}+\omega^2(t)T=C^2/T^3 \tag{34}$$

$$\dot{\xi}T^2=C \tag{35}$$

with C being a real positive constant. One can show then that the last two equations describe the amplitude $T(t)$ and phase $\xi(t)$ of a classical time-dependent oscillator.

Equation (34) has the form of the Milne–Pinney equation [16] and it can be related in some way [17] to a Hill-like form [18].

Let us note that the constant C in (34) and (35) can be absorbed in T letting $T \to C^{1/2}T$ and for $\omega(t)=\omega_0$ and the choice $C=m$ we get from (34) and (35) $T=(m/\omega_0)^{1/2}$ and $\xi=\omega_0 t$. Introducing these quantities into (33) we reduce it to the solution (32).

5. Concluding remarks

The present work extends the class [6] of exactly solvable quantum mechanical problems with time-dependent boundary conditions. The models proposed in section 3.1 and 4 case A are solved for given particular functions $L(t)$ whereas those in section 3.2 and 4 case B admit any choice of $L(t)$. In each case, apart from the usual 'dynamical' phase factors $\exp((-i/\hbar)\Lambda_n t)$, $\exp((-i/\hbar)\lambda_n t)$, $\exp[-i\omega_0(2n+\frac{3}{2})t]$ and $\exp[-i(2n+\frac{3}{2})\xi(t)]$ equations (20), (27), (32) and (33) contain a number of additional phase factors generated by time-dependent boundaries. Clearly, they appear even for a non-adiabatic and non-cyclic evolution.

The phases with no coordinate dependence involved produce only shifts to the above 'dynamical' phase factors. Thus, moving boundaries, being a source of an additional energy to the systems under consideration, modify the eigenvalues of the Hamiltonians H_a and H_b defined in (8). As a result, we arrive at an effective dynamical phase factor of the form $\exp[-(i/\hbar)\int E_k(t)\,dt]$, $k=1,2,3,4$.

Except for this factor the four solutions contain some coordinate-dependent phases reflecting the influence of a moving boundary on the wavefunction of the quantum particle even though the particle has never been in the vicinity of the boundary. This is a non-local effect, not fully comprehensible so far [7].

A similar modification of the phases of wavefunctions has been observed during the passage of a particle through a bottleneck [19] or a tube of a given length [20] or by the moving wall of an infinite square well [21].

A deeper understanding of the coordinate-dependent phases could probably be gained if they were related to the Berry phase and its generalizations. A modest step in that direction has recently been made [22].

As a final point we shall comment on possible applications of the solutions found.

The functions (27), (32) and (33) can be used for studying the response of quantum systems to periodic perturbations. Though the classical counterparts of the three models do not lead to chaotic solutions, as we show elsewhere [7] such quantum models themselves can be a rich source of interesting effects.

The most valuable is the bouncer model of section 3.1. Classically, it is known to show all the types of behaviour from regular to chaotic. Note that (20) represents an exact wavefunction within a period of oscillation of the moving platform of the bouncer. The solution (20) can be used for the construction of an evolution matrix for one period with a part of the parabola (17) describing the platform's move. A repeated application of the matrix (say, n times) will evolve the system through n periods. Thus, we have an exactly solvable quantum model that can be utilized for studying the properties of quantum systems of which classical mechanics can be chaotic. This is currently investigated.

We should add for completeness that perhaps the first model found which allows one to calculate exactly the evolution matrix within one period was the Fermi–Ulam model with the function $L(t)=(1+2at)^{1/2}$ [23, 24]. The same feature has its variant [6] with $L(t)=(at^2-2bt+d)^{1/2}$ and $ad-b^2=c^2$, $c>0$. Though for the latter case an exact wavefunction for one period is known [6], it is much more difficult to use than the solution of the bouncer model introduced here.

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